Final

2024 Canadian Open Mathematics Challenge

Canadian Mathematical Society
Société mathématique du Canada

Official Competition Booklet

Please completely fill the appropriate circles:

Will you be 19 years old <u>or under</u> on June 30, 2025? Ⓨ Ⓝ

Why we ask this: The COMC has an age limit for official participation matching next summer's IMO.

Are you a Canadian Citizen or Permanent Resident of Canada? Ⓨ Ⓝ

Why we ask this: We need to know which of our two primary divisions you qualify in: the Canadian Division or the Foreign/International Division.

Your email address: ______________________________

Why we ask this: If your score on the COMC is high enough, we'll want to contact you for higher level invitation-only competitions such as the Canadian Mathematical Olympiad. ***Please print clearly!***

DO NOT PHOTOCOPY BLANK EXAMS
Each page has a unique pre-registered bar code corresponding to a particular student.

Question A1 (4 points)

Two locations A and B are connected by a 5-mile trail which features a lookout C. A group of 15 hikers started at A and walked along the trail to C. Another group of 10 hikers started at B and walked along the trail to C. The total distance travelled to C by all hikers from the group that started in A was equal to the total distance travelled to C by all hikers from the group that started in B.

Find the distance (in miles) from A to C along the trail.

Your solution:

Your final answer:

[A correct answer here earns full marks]

Question A2 (4 points)

Alice and Bob are running around a rectangular building measuring 100 by 200 meters. They start at the middle of a 200 meter side and run in the same direction, Alice running twice as fast as Bob.

After Bob runs one lap around the building, what fraction of the time were Alice and Bob on the same side of the building?

Your solution:

Your final answer:

[A correct answer here earns full marks]

Question A3 (4 points)

Colleen has three shirts: red, green, and blue; three skirts: red, green, and grey; three scarves: red, blue, and grey; and three hats: green, blue, and grey.

How many ways are there for her to pick a shirt, a skirt, a scarf, and a hat, so that two of the four clothes are one color and the other two are one other color?

Your solution:

Your final answer:

[A correct answer here earns full marks]

Question A4 (4 points)

Consider the sequence of consecutive even numbers starting from 0, arranged in a staggered format, where each row contains one more number than the previous row. The beginning of this arrangement is shown below:

$$\begin{array}{ccccc} 0 & & & & \\ 2 & 4 & & & \\ 6 & \underline{8} & 10 & & \\ 12 & 14 & 16 & 18 & \\ 20 & 22 & 24 & 26 & 28 \end{array}$$

The number in the middle of the third row is 8. What is the number in the middle of the 101-st row?

Your solution:

Your final answer:

[A correct answer here earns full marks]

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NO PHOTOCOPIES!

Question B1 (6 points)

For any positive integer number k, the factorial $k!$ is defined as a product of all integers between 1 and k inclusive: $k! = k \times (k-1) \times \cdots \times 1$.
Let $s(n)$ denote the sum of the first n factorials, i.e.

$$s(n) = \underbrace{n \times (n-1) \times ... \times 1}_{n!} + \underbrace{(n-1) \times (n-2) \times ... \times 1}_{(n-1)!} + \cdots + \underbrace{2 \times 1}_{2!} + \underbrace{1}_{1!}$$

Find the remainder when $s(2024)$ is divided by 8.

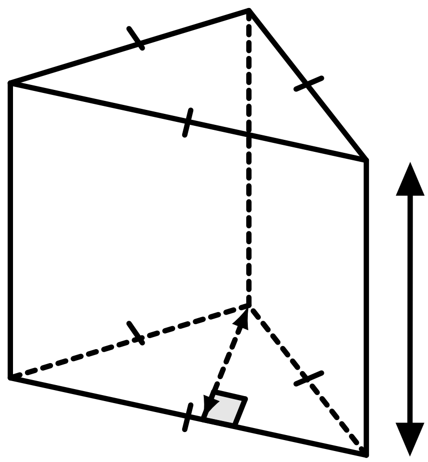
Your solution:

Your final answer:

[A correct answer here earns full marks]

Question B2 (6 points)

David wanted to calculate the volume of a prism with an equilateral triangular base. He was given the height of the prism $H = 15$ and the height of the base $h = 6$. He accidentally swapped the values of H and h in his calculations. By what number should he multiply his result to get the correct volume?



Your solution:

Your final answer:

[A correct answer here earns full marks]

Question B3 (6 points)

Let a, b, c, d be four **distinct** integers such that:

$$\min(a, b) = 2$$
$$\min(b, c) = 0$$
$$\max(a, c) = 2$$
$$\max(c, d) = 4$$

Here $\min(a, b)$ and $\max(a, b)$ denote respectively the minimum and the maximum of two numbers a and b. Determine the fifth smallest possible value for $a + b + c + d$.

Your solution:

Your final answer:

[A correct answer here earns full marks]

Question B4 (6 points)

Initially, the integer 80 is written on a blackboard. At each step, the integer x on the blackboard is replaced with an integer chosen uniformly at random among $[0, x-1]$, unless $x = 0$, in which case it is replaced by an integer chosen uniformly at random among $[0, 2024]$. Let $P(a, b)$ be the probability that after a steps, the integer on the board is b. Determine

$$\lim_{a\to\infty} \frac{P(a, 80)}{P(a, 2024)}$$

(that is, the value that the function $\frac{P(a,80)}{P(a,2024)}$ approaches as a goes to infinity).

Your solution:

Your final answer:

[A correct answer here earns full marks]

Question C1 (10 points)

Let the function $f(x, y, t) = \frac{x^2-y^2}{2} - \frac{(x-yt)^2}{1-t^2}$ for all real values x, y and $t \neq \pm 1$.

(a) Evaluate $f(2, 0, 3)$ and $f(0, 2, 3)$.

(b) Show that $f(x, y, 0) = f(y, x, 0)$ for any values of (x, y).

(c) Show that $f(x, y, t) = f(y, x, t)$ for any values of (x, y) and $t \neq \pm 1$.

(d) Given

$$g(x, y, s) = \frac{(x^2 - y^2)(1 + \sin(s))}{2} - \frac{(x - y\sin(s))^2}{1 - \sin(s)}$$

for all real values x, y and $s \neq \frac{\pi}{2} + 2\pi k$, where k is an integer number, show that $g(x, y, s) = g(y, x, s)$ for any values of (x, y) and s in the domain of $g(x, y, s)$.

Your solution:

You **must** show all your work.

Question C2 (10 points)

(a) How many ways are there to pair up the elements of 1, 2, ..., 14 into seven pairs so that each pair has sum at least 15?

(b) How many ways are there to pair up the elements of 1, 2, ..., 14 into seven pairs so that each pair has sum at least 13?

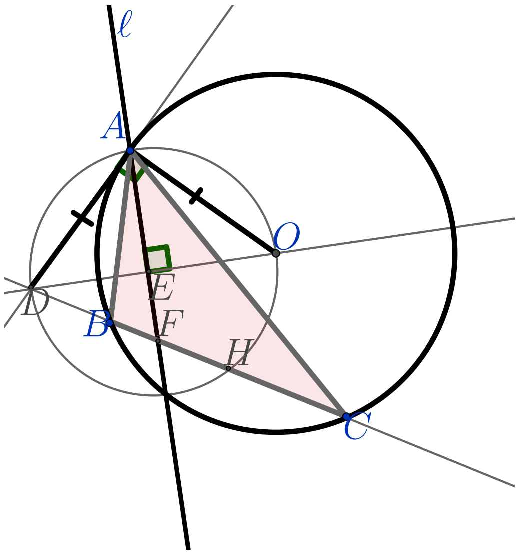
(c) How many ways are there to pair up the elements of 1, 2, ..., 2024 into 1012 pairs so that each pair has sum at least 2001?

Your solution:

You **must** show all your work.

Question C3 (10 points)

Let ABC be a triangle for which the tangent from A to the circumcircle intersects line BC at D, and let O be the circumcenter. Construct the line ℓ that passes through A and is perpendicular to OD. ℓ intersects OD at E and BC at F. Let the circle passing through ADO intersect BC again at H. It is given that $AD = AO = 1$.



(a) Find OE.

(b) Suppose for this part only that $FH = \frac{1}{\sqrt{12}}$: determine the area of triangle OEF.

(c) Suppose for this part only that $BC = \sqrt{3}$: determine the area of the triangle OEF.

(d) Suppose that B lies on the angle bisector of DEF. Find the area of the triangle OEF.

Your solution:

You **must** show all your work.

Question C4 (10 points)

Call a polynomial $f(x)$ *excellent* if its coefficients are all in $[0, 1)$ and $f(x)$ is an integer for all integers x.

(a) Compute the number of excellent polynomials with degree at most 3.

(b) Compute the number of excellent polynomials with degree at most n, in terms of n.

(c) Find the minimum $n \geq 3$ for which there exists an excellent polynomial of the form $\frac{1}{n!}x^n + g(x)$, where $g(x)$ is a polynomial of degree at most $n - 3$.

Your solution:

You **must** show all your work.

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