EGMO 2025

Problems and Solutions

Day 1

P1. For a positive integer N, let $c_1 < c_2 < \cdots < c_m$ be all the positive integers smaller than N that are coprime to N. Find all $N \geq 3$ such that

$$\gcd(N, c_i + c_{i+1}) \neq 1$$

for all $1 \leq i \leq m-1$.

Here $\gcd(a,b)$ *is the largest positive integer that divides both* a *and* b. *Integers* a *and* b *are coprime if* $\gcd(a,b) = 1$.

Solution 1. The answer is all even integers and all powers of 3. First we show that these work.

- When N is even, all c_i are odd, and so $2 \mid \gcd(N, c_i + c_{i+1})$ for every i.
- When N is a power of 3, the c_i are exactly the numbers in the range $1, 2, \ldots, N-1$ that are not divisible by 3. So, the sequence $c_1, \ldots, c_m$ alternates between numbers congruent to $1 \pmod 3$ and $2 \pmod 3$. Thus $3 \mid \gcd(N, c_i + c_{i+1})$ for every i.

Now we show that no other positive integer works. For the sake of contradiction, consider an odd candidate N that is not a power of 3, and suppose it satisfies the problem condition. Then, since $c_1 = 1$ and $c_2 = 2$, we have $\gcd(N, 1+2) \neq 1$, so $3 \mid N$. Thus, we can write N as $3^k M$, where $k \geq 1$ and $\gcd(M, 6) = 1$. We have $M \geq 5$ (as $M \neq 1$ by assumption) and $M + 2 < N$.

We now split into two cases based on M modulo 3.

Case 1. $M \equiv 1 \pmod 3$.

As $3 \mid M - 1$ and $M \mid M$, neither $M-1$ nor M are in the sequence $\{c_i\}$. As M is odd and $M \equiv 1 \pmod 3$, $M-2$ and $M+1$ are congruent to $1 \pmod 3$, are coprime to M, and are smaller than N. Thus, both $M-2$ and $M+1$ are in the sequence, and they are consecutive terms. However, this means $\gcd(N, 2M-1) \neq 1$. This gives a contradiction, as both 3 and M are coprime to $2M-1$.

Case 2. $M \equiv 2 \pmod 3$.

This case is similar to Case 1. Neither M nor $M+1$ are in the sequence, but $M-1$ and $M+2$ are. We obtain a similar contradiction from $\gcd(N, 2M+1) \neq 1$.

Solution 2. We give an alternative way to show that any odd N has to be the power of 3. Suppose for contradiction that N has at least 2 distinct prime factors. Choose integers $a, b > 1$ such that

- $ab \mid N$;
- a and b are coprime;
- Every prime divisor of N divides ab.

We use the Chinese Remainder Theorem, to find an integer $n \in [1, ab]$ such that

$$\begin{cases} n \equiv 1 \pmod a, \\ n \equiv 2 \pmod b. \end{cases}$$

We claim that there is an index i such that $c_i = n - 3$ and $c_{i+1} = n$. Indeed, we note the following:

- n is consecutive to a multiple of a, so $\gcd(a, n) = 1$. In a similar way we get that $\gcd(b, n-3) = 1$.
- n is 2 away from a multiple of b, therefore as $2 \nmid b$, $\gcd(b, n) = 1$. In a similar way we get that $\gcd(a, n-3) = 1$.
- $n \le ab \le N$ and $n - 3 \ge (a + 1) - 3 > 0$ as $a > 2$.

This implies that n and $n - 3$ coprime with N, whereas neither $n - 2$ nor $n - 1$ are coprime with N, which proves our claim. Finally, we claim that $c_i + c_{i+1} = (n - 3) + n = 2n - 3$ is coprime with N. Indeed we have $2n - 3 \equiv -1 \pmod{a}$ and $2n - 3 \equiv 1 \pmod{b}$. This implies that N is a prime power.

When N is odd, since $c_1 = 1$ and $c_2 = 2$, we have $\gcd(N, 1 + 2) \neq 1$, so $3 \mid N$, which implies that N is necessarily a power of 3. This concludes our proof.

P2. An infinite increasing sequence $a_1 < a_2 < a_3 < \cdots$ of positive integers is called *central* if for every positive integer n, the arithmetic mean of the first a_n terms of the sequence is equal to a_n.

Show that there exists an infinite sequence $b_1, b_2, b_3, \ldots$ of positive integers such that for every central sequence $a_1, a_2, a_3, \ldots$, there are infinitely many positive integers n with $a_n = b_n$.

Solution 1. We claim that the sequence b_1, b_2, $b_3, \ldots$ defined by $b_i = 2i - 1$ has this property.

Let $d_i = a_i - b_i = a_i - 2i + 1$. The condition $a_i < a_{i+1}$ now becomes $d_i + 2i - 1 < d_{i+1} + 2i + 1$, which can be rewritten as $d_{i+1} \geqslant d_i - 1$. Thus, if $d_{i+1} < d_i$, then d_{i+1} must be equal to $d_i - 1$. This implies in particular that if $d_{i_0} \geqslant 0$ but $d_{i_1} \leqslant 0$ for some indices $i_1 > i_0$, there must be some intermediate index $i_0 \leqslant i \leqslant i_1$ with $d_i = 0$.

Because the average of the first a_n terms of the sequence is equal to a_n, we know for all n that

$$\sum_{i=1}^{a_n} d_i = \sum_{i=1}^{a_n} (a_i - 2i + 1) = \sum_{i=1}^{a_n} a_i - \sum_{i=1}^{a_n} (2i - 1) = a_n^2 - a_n^2 = 0.$$

Because the sequence (a_n) is increasing, this implies that the sequence (d_i) contains infinitely many non-negative ($d_i \geqslant 0$) and infinitely many non-positive ($d_i \leqslant 0$) terms. In particular, we can find arbitrarily large indices $i_0 \leqslant i_1$ such that $d_{i_0} \geqslant 0$ and $d_{i_1} \leqslant 0$. By our earlier observation, it follows that there are infinitely many i such that $d_i = 0$, as desired.

Solution 2. We give an alternative proof that the sequence $b_i = 2i - 1$ works. This proof is by contradiction, so we assume that there are only finitely many a_i such that $a_i = 2i - 1$.

Let $S(n) = \sum_{i=1}^{n} a_i$. We have $S(a_n) = a_n^2$ and $S(a_{n+1}) = a_{n+1}^2$. If $a_{n+1} = a_n + 1$, then it follows that

$$S(a_{n+1}) - S(a_n) = a_{n+1}^2 - a_n^2 = a_{n+1}^2 - (a_{n+1} - 1)^2 = 2a_{n+1} - 1.$$

On the other hand, if $a_{n+1} = a_n + 1$, then $S(a_{n+1}) - S(a_n)$ is $a_{a_{n+1}}$, so it follows that $a_{a_{n+1}} = 2a_{n+1} - 1$. By assumption, this can only happen finitely many times, so for all sufficiently large n we must have $a_{n+1} \geqslant a_n + 2$.

For large enough n, we now know that $a_n > 2n - 1$ implies $a_{n+1} > (2n - 1) + 2 = 2(n + 1) - 1$. This means that there are two cases possible:

(A) For all sufficiently large n (say $n \geqslant N_A$) we have $a_n > 2n - 1$.

(B) For all sufficiently large n (say $n \geqslant N_B$) we have $a_n < 2n - 1$.

In case (A), we know for $m > N_A$ that

$$\begin{aligned} S(m) = S(N_A) + \sum_{i=N_A+1}^{m} a_i \geqslant S(N_A) + \sum_{i=N_A+1}^{m} 2i &= S(N_A) + m(m+1) - N_A(N_A + 1) \\ &= m^2 + m + S(N_A) - N_A(N_A + 1). \end{aligned}$$

For m large enough (e.g. $m > N_A(N_A + 1)$), this expression is always larger than m^2, contradicting $S(a_n) = a_n^2$ for all n.

Similarly, in case (B), we similarly know for $m > N_B$ that

$$\begin{aligned} S(m) = S(N_B) + \sum_{i=N_B+1}^{m} a_i \leqslant S(N_B) + \sum_{i=N_B+1}^{m} 2(i-1) &= S(N_B) + m(m-1) - N_B(N_B - 1) \\ &= m^2 - m + S(N_B) - N_B(N_B - 1). \end{aligned}$$

For m large enough (e.g. $m > S(N_B)$), this expression is always smaller than m^2, again contradicting $S(a_n) = a_n^2$ for all n.

Solution 3. We claim that the sequence $b_1, b_2, b_3, \ldots$ defined by $b_i = 2i - 1$ has this property.

Lemma. *If there are no terms a_j such that $a_j - a_{j-1} = 1$, then $a_j = a_{j-1} + 2$ for all j.*

Proof. Let c be such that $a_d = c$ for some d. Now

$$a_1 + a_2 + \cdots + a_c = c^2.$$

Equality holds for $a_i = 2i - 1$ for $1 \leq i \leq c$, so if any difference between two consecutive terms is greater, the left-hand side of the equation is greater than c^2, a contradiction. □

Lemma. *If both d and $d+1$ are terms of the sequence, i.e. $a_c = d$ and $a_{c+1} = d+1$ for some c, then $a_{d+1} = 2d + 1 = b_{d+1}$.*

Proof. We have $a_1 + a_2 + \cdots + a_d = d^2$ and $a_1 + a_2 + \cdots + a_{d+1} = (d+1)^2$. Hence $a_{d+1} = (d+1)^2 - d^2 = 2d + 1$. □

From the observations above, we see that we are done if there are infinitely many gaps of size 1. The only remaining case is one with finitely many gaps of size 1. This will be the subject of the following lemma.

Lemma. *If there are only finitely many indices j such that $a_{j+1} - a_j = 1$, then there is an index n_0 such that for all $k > n_0$, we have $a_k = 2k - 1$.*

Proof. Let r and s be indices such that for all the j satisfying $a_{j+1} - a_j = 1$, we have $j < r, s$. Furthermore, assume $s > r$ and that there are i_1 and i_2 such that $a_{i_1} = r$ and $a_{i_2} = s$. The first goal is to show that $a_s \geqslant 2s - 1$. If $a_r \geqslant 2r - 1$, this is clearly the case. Assume now $a_r < 2r - 1$. Now $a_r \geqslant 2r - 1 - m$, where m is the number of indices j with $a_{j+1} - a_j = 1$. Denote $a_{r+1} = 2r + 1 - m + \theta_1$, $a_{r+2} = 2r + 3 - m + \theta_2$, etc. Remember that $a_{r+j+1} - a_{r+j} \geqslant 2$ always. Now $0 \leqslant \theta_1 \leqslant \theta_2 \leqslant \ldots$. Furthermore, write $s = r + h$. Now

$$(r+h)^2 - r^2 = a_{r+1} + a_{r+2} + \cdots + a_{r+h} = \sum_{j=1}^{h} 2r - 1 + 2j - m + \theta_j.$$

From this we deduce

$$2rh + h^2 = 2rh - h - mh + h(h+1) + \sum_{j=1}^{h} \theta_j.$$

So we obtain $\sum_{j=1}^{h} \theta_j = mh$. Since the sequence θ_j is increasing, we have $\theta_h \geqslant m$. Hence, $a_s = a_{r+h} \geqslant 2r - 1 - m + 2h + m = 2r + 2h - 1 = 2s - 1$.

Now a_s is exactly the desired shape. If for any $t > s$, we have $a_t - a_{t-1} > 2$, then

$$a_s + a_{s+1} + \cdots + a_t > t^2 - s^2,$$

again a contradiction. □

Solution 4. Note that $a_1 = 1$ because if it is not the case, then $a_1^2 = a_1 + \cdots + a_{a_1} > a_1 + a_1 + \cdots + a_1 = a_1^2$.

Assume by contradiction that there are only finitely many indices k such that $a_k = 2k - 1$. Set i to be the largest integer such that $a_i = 2i - 1$ (which must exist as $a_1 = 1$). Assume that there exists $j \geqslant i$ such that $a_{j+1} - a_j = 1$. Then $2a_{j+1} - 1 = a_{j+1}^2 - a_j^2 = a_{a_{j+1}}$ and since $a_k \geqslant k$ for all k, we have $a_{j+1} \geqslant j + 1 > i$, which contradicts the definition of i. Thus for all $j \geqslant i$, we have $a_{j+1} \geqslant a_j + 2$, which implies by induction that $a_j \geqslant 2j - 1$ for $j \geqslant i$, and even $a_j \geq 2j$ if $j > i$.

There are two ways to finish the solution from here.

First way to finish the solution

For all n such that $a_n \geqslant i$, we have

$$\begin{aligned} a_{n+1}^2 - a_n^2 = a_{a_{n+1}} + a_{a_{n+1}-1} + \cdots + a_{a_n+1} &\geqslant 2a_{n+1} + 2(a_{n+1} - 1) + \cdots + 2(a_n + 1) \\ &= (a_{n+1} - a_n)(a_{n+1} + a_n + 1) \\ &> a_{n+1}^2 - a_n^2. \end{aligned}$$

This gives a contradiction.

Second way to finish the solution

For all n such that $a_n \geqslant i$, we introduce $x_n = a_{n+1} - a_n$. We have

$$x_n^2 + 2x_n a_n = a_{n+1}^2 - a_n^2 = a_{a_{n+1}} + a_{a_{n+1}-1} + \cdots + a_{a_n+1} \geqslant \sum_{j=1}^{x_n} (a_{a_n} + 2j) \geq x_n a_{a_n} + x_n(x_n + 1).$$

By simplifying, we get $a_{a_n} \leq 2a_n - 1$, which gives a contradiction.

Comment. Proving that $a_1 = 1$ is not necessary for this solution. If there exists no i such that $a_i = 2i - 1$, then the same argument implies that there exists no j such that $a_{j+1} - a_j = 1$, thus $a_j \geqslant 2j - 1$ for $j \geqslant 1$.

P3. Let ABC be an acute triangle. Points B, D, E, and C lie on a line in this order and satisfy $BD = DE = EC$. Let M and N be the midpoints of AD and AE, respectively. Let H be the orthocentre of triangle ADE. Let P and Q be points on lines BM and CN, respectively, such that D, H, M, and P are concyclic and E, H, N, and Q are concyclic. Prove that P, Q, N, and M are concyclic.

The orthocentre of a triangle is the point of intersection of its altitudes.

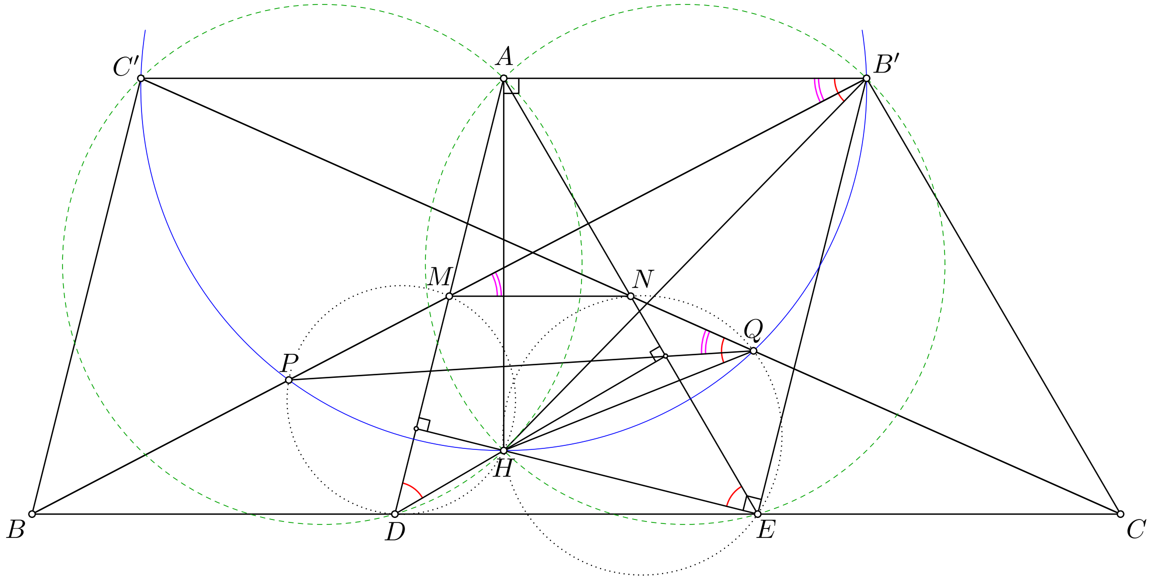
Solution 1. Denote by B' and C' the reflections of B and C in M and N, respectively. Points C', A, B' are clearly collinear and $DEB'A$ is a parallelogram. Since $EH \perp AD$, we have $EH \perp EB'$. Also $HA \perp AB'$, so points H, E, B', A are concyclic. This gives

$$\angle C'QH = \angle NQH = \angle NEH = \angle AEH = \angle AB'H = \angle C'B'H,$$

and so points C', B', Q, H are concyclic. Analogously points C', B', P, H are concyclic, and so all points B', C', P, Q, H are. Now we have

$$\angle NMB' = \angle AB'M = \angle C'B'P = \angle C'QP = \angle NQP,$$

which proves that P, Q, N, M are also concyclic.



Solution 1′. Introduce points B', C' as above. Also define $A' = B'E \cap C'D$, so that $EADA'$ is a parallelogram and ADE is the medial triangle of $A'B'C'$. It that follows that the orthocentre H of ADE is the circumcentre of $A'B'C'$, and in particular

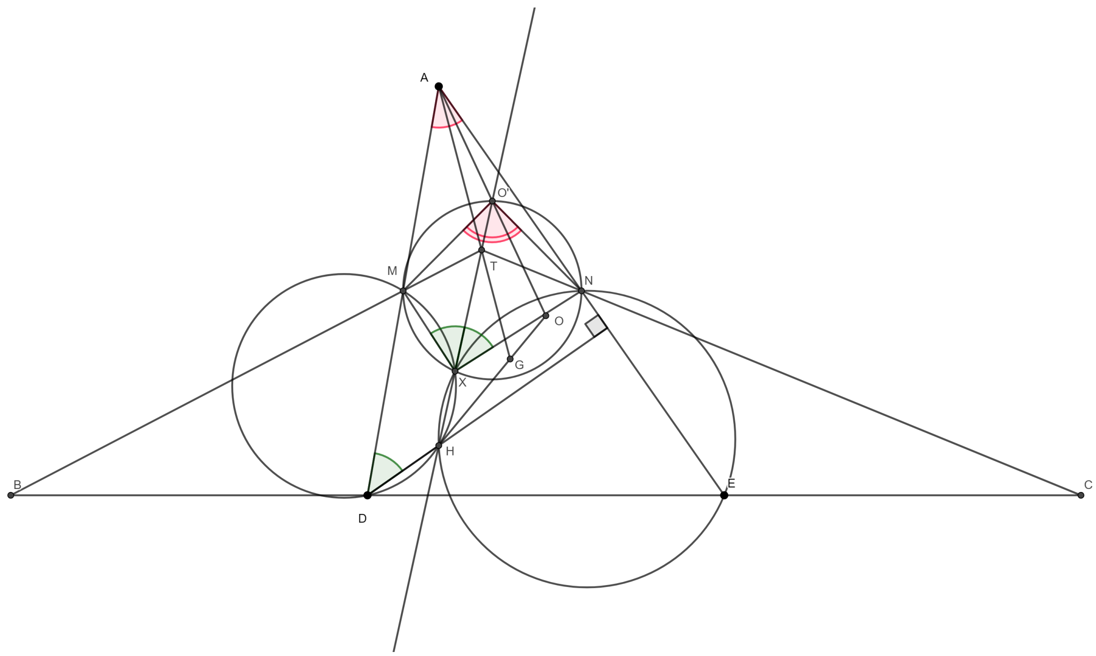
$$\angle C'B'H = 90^\circ - \angle B'A'C' = 90^\circ - \angle DAE = \angle AEH = \angle NEH = \angle NQH = \angle C'QH.$$

So again we have that C', B', Q, H are concyclic and conclude as in Solution 1.

Solution 2. Let X be the second intersection of (DHM) and (EHN) and let O' be the circumcentre of (AMN). Note that $\angle MXN = \angle MDH + \angle NEH = 180^\circ - 2\angle DAE$ and since $\angle MO'N = 2\angle DAE$ we have that X, M, O', N is cyclic and since $\angle MXH = \angle NXH$ it means that HX is the angle bisector of $\angle MXN$ but since $O'M = O'N$ it means that H, X, O' are collinear. Let BM and CN intersect at T and let K and L be the midpoints of MN and BC. Note that L is also the midpoint of DE. Since MN is parallel to BC it means that T, K, and L are collinear, but since A, K, and L are collinear we get that A, T, K, and L are collinear. Now, $\frac{TL}{TK} = \frac{BC}{MN} = 6$. Since $KL = KA$ it means that $\frac{AT}{TK} = \frac{TL-2TK}{TK} = 4$ so by the lemma below, T lies on HO'. Since HO' is the radical axis of (DHM) and (EHN) we finish the problem using the Radical Axes Theorem ($TM \cdot TP = TN \cdot TQ$).

Lemma. *Let A' be the reflection of A around the orthocentre H of $\triangle ABC$ and O and M be the circumcentre of $\triangle ABC$ and the midpoint of BC, respectively. Let T be the intersection of $A'O$ and AM. Then $\frac{AT}{TM} = 4$.*

Proof. Since $OM \parallel AA'$ we have $\frac{AT}{TM} = \frac{AA'}{OM} = \frac{2AH}{OM} = \frac{4OM}{OM} = 4$. We used here that $AH = 2OM$. $\square$



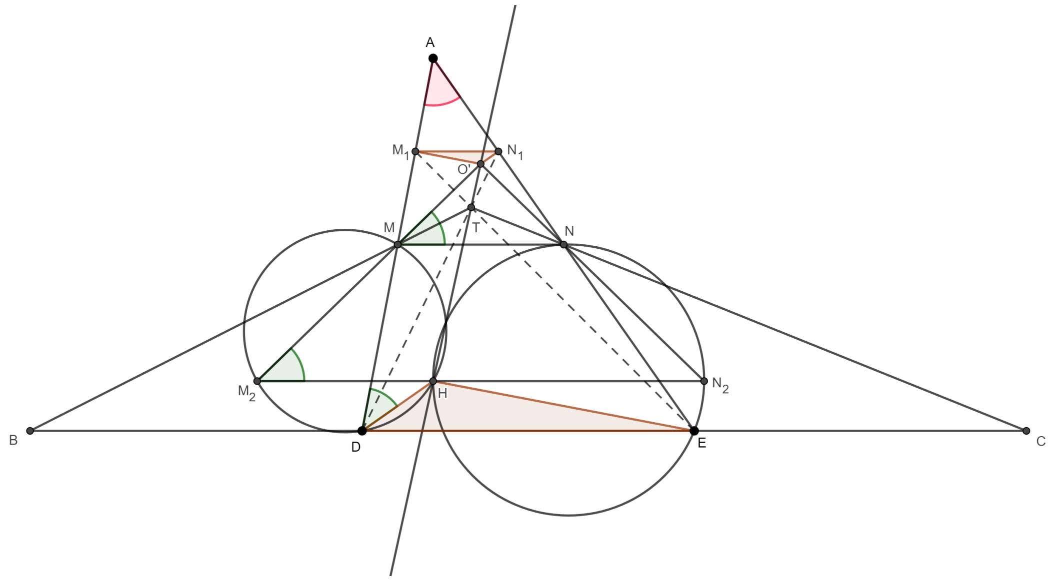
Solution 2′. As in solution 2, we will prove that O' is both on line HT and the radical axis of the circles, hence T is on the radical axis, from which we conclude the required concyclicity. We present alternative proofs of both facts, discovered by contestants.

Let M_1, N_1 be the midpoints of AM, AN, respectively, so that $AM_1 : M_1D = AN_1 : N_1E = 1 : 3$. It is easy to verify (e.g. by applying the theorems of Ceva or Menelaus, or by computing in barycentric coordinates as in Solution 3) that T lies on EM_1 and DN_1. Note that $M_1N_1 \parallel DE$, and also $M_1O' \parallel HE$ as both are perpendicular to AMD, and similarly $N_1O' \parallel HD$. It follows that DEH and N_1M_1O' are homothetically similar triangles, and the center of their (negative) homothety is $T = DN_1 \cap EM_1$. Therefore T also lies on HO', as claimed. (We also have that the homothety is by factor $\frac{M_1N_1}{ED} = -\frac{1}{4}$.)

Now, let M_2, N_2 be the second intersection points of $O'M$, $O'N$ with the circumcircles of HMD, HNE, respectively. To prove that O' is on the radical axis, it suffices to show that $O'M \cdot O'M_2 = O'N \cdot O'N_2$. But by definition of O' we have $O'M = O'N$, so we must show $O'M_2 = O'N_2$, which is equivalent to $M_2N_2 \parallel MN$. Angle chasing in circle MM_2DH gives

$$\angle OM_2H = \angle MM_2H = \angle MDH = \angle ADH = 90^\circ - \angle EAD = 90^\circ - \angle NAM = \angle O'MN,$$

from which it follows that $M_2H \parallel MN$. Similarly, we have $N_2H \parallel MN$, and the two facts together imply that M_2, H, N_2 are collinear and the line through them is parallel to MN, as claimed. $\square$



Solution 3. We compute using linear combinations with respect to ADE. We have $B = 2D - E$, $C = 2E - D$, $M = \frac{A+D}{2}$, and $N = \frac{A+E}{2}$, from which we immediately obtain that the intersection of $T = BM \cap CN$ is $T = \frac{3A+D+E}{5} = \frac{6M-B}{5} = \frac{6N-C}{5}$. As in solution 2, we reduce to showing that T is on the radical axis of HDM and HEN, whence $TM \cdot TP = TN \cdot TQ$ proves the concyclicity of P, Q, N, M.

Synthetic finish, similar to Solution 2. Let O be the circumcentre of ADE and $O' = \frac{A+O}{2}$ be the circumcentre of AMN. As in Solution 2, we have that O' is on the desired radical axis, so it is enough to show $T \in O'H$. Let $G = \frac{A+D+E}{3}$ be the barycentre of ADE. By properties of the Euler line, we also have $G = \frac{H+2O}{3}$. Now using our known identities we find

$$T = \frac{3A+D+E}{5} = \frac{2A+3G}{5} = \frac{2A+H+2O}{5} = \frac{H+4O'}{5}$$

and in particular $T \in HO'$, as we wanted to show.

Computational finish. Let $f(T)$ be the power difference at T w.r.t. DHM and EHN. We compute $f(A)$ and $f(L)$ where $L = \frac{D+E}{2}$. Since $T = \frac{3A+2L}{5}$, it is enough to show that $3P(A) + 2P(L) = 0$. In the following all lengths are directed. We compute trigonometrically: Let α, β, γ be the angles of ADE and assume the diameter of its circumcircle is 1. We have

$$f(A) = \frac{AD^2 - AE^2}{2} = \frac{\sin^2(\gamma) - \sin^2(\beta)}{2}.$$

To compute $P(L)$, let D', E' be the second intersection points of HMD, HNE with DE, and let M', N', F be the feet of the perpendiculars from H to AD, AE, DE, respectively. Note that $DM' = \sin\alpha\cos\beta$, thus

$$M'M = DM - DM' = \frac{\sin(\alpha+\beta)}{2} - \sin\alpha\cos\beta = \frac{\sin(\beta-\alpha)}{2}.$$

We also have $HM' = \cos\alpha\cos\beta$, $HF = \cos\beta\cos\gamma$, and $HM'M \sim HFD'$, therefore

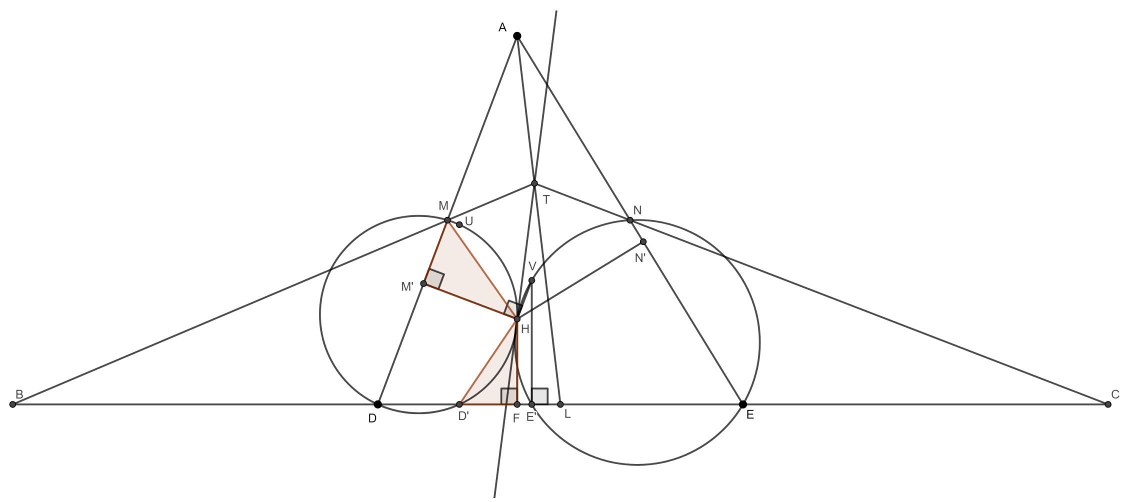
$$FD' = \frac{HF}{HM'}M'M = \frac{\cos\gamma}{\cos\alpha}\frac{\sin(\beta-\alpha)}{2}$$

and similarly

$$E'F = \frac{\cos\beta}{\cos\alpha}\frac{\sin(\gamma-\alpha)}{2}.$$

We also have the standard $FD = \sin\gamma\cos\beta$ and $EF = \sin\beta\cos\gamma$. We can finally compute

$$\begin{aligned}
f(L) &= LD\cdot LD' - LE\cdot LE' = \frac{\sin\alpha}{2}(LD' + LE') = \frac{\sin\alpha}{2}(FD' + FE' - FD - FE)\\
&= \frac{\sin\alpha}{4\cos\alpha}(\cos\gamma\sin(\beta-\alpha) - \cos\beta\sin(\gamma-\alpha) - 2\cos\alpha(\sin\gamma\cos\beta - \sin\beta\cos\gamma))\\
&= \frac{3\sin\alpha\sin(\beta-\gamma)}{4} = \frac{3}{4}(\sin(\beta+\gamma)\sin(\beta-\gamma)) = \frac{3}{8}(\cos(2\gamma) - \cos(2\beta))\\
&= \frac{3}{4}(\sin^2(\beta) - \sin^2(\gamma)) = -\frac{3}{2}f(A). \quad \square
\end{aligned}$$



Solution 4. Let $T = BM \cap CN$, let F be the foot of the altitude from A to BC, let O be the circumcentre of (ADE), let $D' \neq D$ and $E' \neq E$ be the second intersections of $(DHMP)$ and $(EHNQ)$ with line BC, and let U and V be the antipodes of D and E on $(DHMP)$ and $(EHNQ)$, respectively.

We begin with a bit of barycentric coordinates. Set barycentric coordinates in $\triangle ABC$, set so that $A = (1, 0, 0)$, $B = (0, 1, 0)$, and $C = (0, 0, 1)$. The definitions of D and E give $D = (0, 2/3, 1/3)$ and $E = (0, 1/3, 2/3)$, whence $M = (1/2, 1/3, 1/6)$ and $N = (1/2, 1/6, 1/3)$. This means that line BM is given by $(1/2 : y : 1/6)$ while line CN is given by $(1/2 : 1/6 : z)$. So their intersection T is $(1/2 : 1/6 : 1/6) = (3 : 1 : 1)$, giving $T = \frac{3A+B+C}{5} = \frac{3A+D+E}{5}$.

Our next tool is the linearity of the power of a point. Let $f\colon \mathbb{R}^2 \to \mathbb{R}$ be defined by

$$f(Z) = \operatorname{Pow}_{(DHMP)}(Z) - \operatorname{Pow}_{(EHNQ)}(Z).$$

It suffices to show that $f(T) = 0$; from there, the required concyclicity will follow from $TM \cdot TP = TN \cdot TQ$. The key observation is that f is a linear function. In particular, $f(T) = \frac{3f(A)+f(D)+f(E)}{5}$. So, we need only show that $3f(A) + f(D) + f(E) = 0$. Pick an arbitrary one-dimensional coordinate system on the line BC and let τ be the map from a point on BC to its coordinate. We compute

$$\begin{aligned}
f(A) &= AM\cdot AD - AN\cdot AE = \frac{AD^2 - AE^2}{2} = \frac{FD^2 - FE^2}{2}\\
&= (\tau(E) - \tau(D))\left(\tau(F) - \tau\left(\frac{D+E}{2}\right)\right),\\
f(D) &= -DE\cdot DE' = (\tau(E) - \tau(D))(\tau(D) - \tau(E')),\\
f(E) &= ED\cdot ED' = (\tau(E) - \tau(D))(\tau(E) - \tau(D')).
\end{aligned}$$

Rearranging, it suffices to show that $3\tau(F) = \tau(D') + \tau(E') + \tau((D+E)/2)$. This can be rewritten as $3F = D' + E' + (D+E)/2$. By projecting down to line BC, it suffices to show that the displacement vector $H + 2A - (O + U + V)$ is perpendicular to line BC.

We do this using complex numbers. Let (ADE) be the unit circle with $A = a$, $D = d$, and $E = e$, so that $O = 0$ and $H = h = a + d + e$. Note that U satisfies $UM \perp AD$ and $UH \perp DH \perp AE$. Translating these conditions into equations, we have $u = ad\overline{u}$ and $u + ae\overline{u} = h + ae\overline{h}$. Rearranging gives

$$(d+e)u = du + e(ad\overline{u}) = d(h + ae\overline{h}) \implies u = \frac{dh + ade\overline{h}}{d+e}.$$

Computing v similarly, we get that

$$v := H + 2A - (O + U + V) = h + 2a - \frac{(d+e)h + 2ade\overline{h}}{d+e} = 2\left(a - \frac{ade\overline{h}}{d+e}\right) = -\frac{2de}{d+e}.$$

This displacement vector v satisfies $v = de\overline{v}$ and so it is orthogonal to line DE, as desired.

Solution 5. This solution uses almost exclusively complex numbers. As in other solutions, we reduce to showing that H, $S := (DHM) \cap (EHN)$, and $T := BM \cap CN$ are collinear; this is all of the synthetic information we shall use. (If one computes $T = \frac{3A+D+E}{5}$ using means other than complex numbers, the solution becomes much shorter.)

We use complex numbers with $A = a$, $D = d$, and $E = e$ on the unit circle. Note that $H = a + d + e$, $M = \frac{a+d}{2}$, and $B = 2d - e$. We will make great use of the "arbitrary line intersection formula," which says that the intersection between lines WX and YZ can be written explicitly as

$$\frac{(\overline{w}x - w\overline{x})(y - z) - (w - x)(\overline{y}z - y\overline{z})}{(\overline{w} - \overline{x})(y - z) - (w - x)(\overline{y} - \overline{z})}.$$

We first use this to find $T = t$. We compute

$$\begin{aligned}
b - m &= (2d - e) - \frac{a+d}{2} = \frac{3d - a - 2e}{2} \\
\overline{b - m} &= \frac{3ae - de - 2ad}{2ade} \\
\overline{b}m - b\overline{m} &= \frac{2e - d}{de} \cdot \frac{a+d}{2} - (2d - e) \cdot \frac{a+d}{2ad} \\
&= \frac{a(a+d)(2e-d) - (2d-e)(a+d)e}{2ade} = \frac{(a+d)(2ae + e^2 - ad - 2de)}{2ade}.
\end{aligned}$$

If $\mathcal{E}$ is some expression, we use the notation $\mathcal{E} - \{\sim\}$ to denote $\mathcal{E}$ minus the expression formed by swapping d and e in $\mathcal{E}$. We now have

$$\begin{aligned}
t &= \frac{(\overline{b}m - b\overline{m})(c - n) - \{\sim\}}{(\overline{b} - \overline{m})(c - n) - \{\sim\}} \\
&= \frac{(a+d)(2ae + e^2 - ad - 2de)(3e - a - 2d) - \{\sim\}}{(3ae - de - 2ad)(3e - a - 2d) - \{\sim\}} \\
&= \frac{\left[a^3(d - 2e) + a^2(3d^2 - 7de + 5e^2) + a(2d^3 - d^2e - 3de^2 + 3e^3) + de(4d^2 - 8de + 3e^2)\right] - \{\sim\}}{\left[a^2(2d - 3e) + a(4d^2 - 11de + 9e^2) + de(2d - 3e)\right] - \{\sim\}} \\
&= \frac{a^3(3(d-e)) - a^2(2(d^2 - e^2)) + a(-(d^3 - e^3) + 2de(d - e)) + de(d^2 - e^2)}{a^2(5(d-e)) - a(5(d^2 - e^2)) + de(5(d-e))} \\
&= \frac{3a^3 - 2a^2(d+e) - a(d^2 - de + e^2) + de(d+e)}{5\bigl(a^2 - a(d+e) + de\bigr)} \\
&= \frac{(a-d)(a-e)(3a + d + e)}{5(a-d)(a-e)} = \frac{3a + d + e}{5}.
\end{aligned}$$

(The factorization in the last line can be motivated by noting that the expression, while cubic in a, is only quadratic in d. When written out as a polynomial in d, each coefficient is divisible by $a - e$; by symmetry, the numerator is divisible by $a - d$ as well, and the factorization follows.)

Computing $S = s$ is slightly harder, as it is the intersection of two circles rather than of two lines. We get around this by noting that $\{h, d, m, s\}$ are concyclic if and only if $\{0, h-d, h-m, h-s\}$ are concyclic, which happens if and only if $\{\frac{1}{h-d}, \frac{1}{h-m}, \frac{1}{h-s}\}$ are collinear. (One can see this by inversion, or just by writing out the cross-ratio in the special case when one of the points is zero.) Thus $\frac{1}{h-s}$ is the intersection of the line through $w := \frac{1}{h-d}$ and $x := \frac{1}{h-m}$ and the line through $y := \frac{1}{h-e}$ and $z := \frac{1}{h-n}$. We compute

$$\begin{aligned}
w - x &= \frac{1}{a+e} - \frac{1}{\frac{a+d}{2}+e} = -\frac{a-d}{(a+e)(a+d+2e)} \\
\overline{w-x} &= \frac{ae^2(a-d)}{(a+e)(2ad+ae+de)} \\
\overline{w}x - w\overline{x} &= \frac{ae}{a+e} \cdot \frac{2}{a+d+2e} - \frac{1}{a+e} \cdot \frac{2ade}{2ad+ae+de} \\
&= 2ae\frac{(2ad+ae+de) - d(a+d+2e)}{(a+e)(a+d+2e)(2ad+ae+de)} \\
&= \frac{2ae(a-d)(d+e)}{(a+e)(a+d+2e)(2ad+ae+de)}.
\end{aligned}$$

Using the line intersection formula, we have

$$\begin{aligned}
\frac{1}{h-s} &= \frac{(\overline{w}x - w\overline{x})(y-z) - \{\sim\}}{(\overline{w}-\overline{x})(y-z) - \{\sim\}} \\
&= \frac{\left[-\frac{2ae(a-d)(d+e)}{(a+e)(a+d+2e)(2ad+ae+de)} \cdot \frac{(a-e)}{(a+d)(a+2d+e)}\right] - \{\sim\}}{\left[-\frac{ae^2(a-d)}{(a+e)(2ad+ae+de)} \cdot \frac{(a-e)}{(a+d)(a+2d+e)}\right] - \{\sim\}} \\
&= 2(d+e)\frac{\left[e(ad+2ae+de)\right] - \{\sim\}}{\left[e^2(a+d+2e)(ad+2ae+de)\right] - \{\sim\}} \\
&= 2(d+e)\frac{\left[a(de+2e^2)+de^2\right] - \{\sim\}}{\left[a^2(de^2+2e^3) + a(d^2e^2+5de^3+4e^4) + de(de^2+2e^3)\right] - \{\sim\}} \\
&= \frac{2(d+e)(a(2(e^2-d^2)) + de(e-d))}{(a^2+de)(2(e^3-d^3)+de(e-d)) + a(4(e^4-d^4)+5de(e^2-e^2))} \\
&= \frac{2(d+e)(2a(d+e)+de)}{(a^2+de)(2d^2+3de+2e^2) + a(d+e)(4d^2+5de+4e^2)}.
\end{aligned}$$

Since $h - t = \frac{2a+4d+4e}{5}$, this gives us

$$\frac{h-s}{h-t} = \frac{5}{4} \cdot \frac{(a^2+de)(2d^2+3de+2e^2) + a(d+e)(4d^2+5de+4e^2)}{(d+e)(2ad+2ae+de)(a+2d+2e)}.$$

It is easy to see that this is real by using the symmetry of the expressions (both the numerator and denominator satisfy $\mathcal{E} = a^2d^3e^3\overline{\mathcal{E}}$. We conclude that H, S, and T are collinear, as desired.

Solution 6. As usual, we reduce to proving $T = (3A + D + E)/5$ is on the radical axis and compute; this time in Cartesian coordinates.

Let $A(0, h)$, $D(b, 0)$, $E(c, 0)$ be coordinates for ADE. Then $H(0, -\frac{bc}{h})$, $M(\frac{b}{2}, \frac{h}{2})$, $N(\frac{c}{2}, \frac{h}{2})$ and $T(\frac{b+c}{5}, \frac{3h}{5})$. We compute $O_D(x_D, y_D)$ the circumcentre of DMH and obtain O_E by symmetry. We then have to verify that $O_DO_E \perp HT$, which can be done by comparing slopes.

The centre O_D can be given by the intersection of perpendicular bisectors of DM and DH. This gives the following system of equations on x_D, y_D:

$$h \cdot x_D + c \cdot y_D = \frac{bh}{2} - \frac{bc^2}{2h}$$
$$-b \cdot x_D + h \cdot y_D = -\frac{3b^2}{4} + \frac{h^2}{4}$$

Solving the system gives

$$(h^2 + bc)x_D = \frac{3b^2c}{4} - \frac{bc^2}{2} + \frac{h^2b}{2} - \frac{h^2c}{4}$$
$$(h^2 + bc)y_D = -\frac{hb^2}{4} + \frac{h^3}{4} - \frac{b^2c^2}{2h}$$

The formulas for x_E, y_E will be the same, swapping $b \leftrightarrow c$ by symmetry; thus $y_D - y_E$ and $x_D - x_E$ will be antisymmetric in b, c and divisible by $b - c$, and explicitly:

$$(h^2 + bc)(x_D - x_E) = \frac{b - c}{4}(5bc + 3h^2)$$
$$(h^2 + bc)(y_D - y_E) = -\frac{b - c}{4}h(b + c)$$

So $\frac{y_D - y_E}{x_D - x_E} = -\frac{h(b+c)}{5bc+3h^2}$.
The other slope is more immediate:

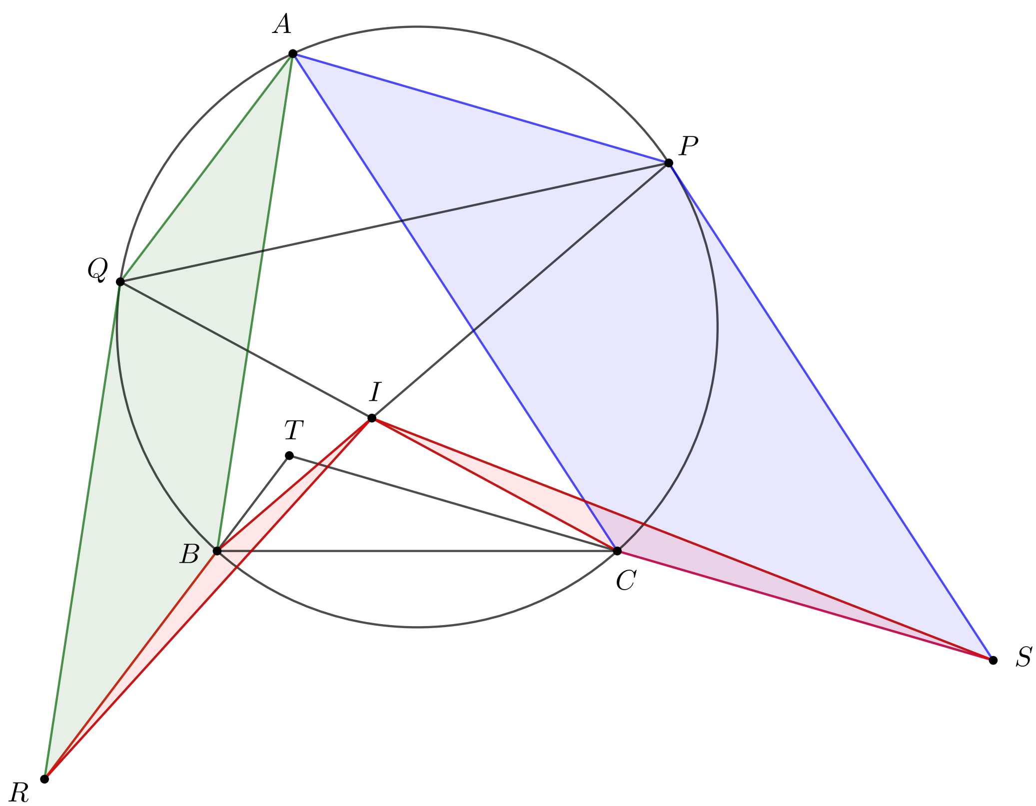
$$\frac{y_T - y_H}{x_T - x_H} = \frac{3h/5 + bc/h}{(b + c)/5} = \frac{5bc + 3h^2}{h(b + c)} = -\frac{x_D - x_E}{y_D - y_E}$$

so indeed the two slopes correspond to perpendicular lines.

Day 2

P4. Let ABC be an acute triangle with incentre I and $AB \neq AC$. Let lines BI and CI intersect the circumcircle of ABC at $P \neq B$ and $Q \neq C$, respectively. Consider points R and S such that $AQRB$ and $ACSP$ are parallelograms (with $AQ \parallel RB$, $AB \parallel QR$, $AC \parallel SP$, and $AP \parallel CS$). Let T be the point of intersection of lines RB and SC. Prove that points R, S, T, and I are concyclic.

Solution 1. We will prove that $\triangle BIR \sim \triangle CIS$, since the statement then follows from $\angle TRI = \angle BRI = \angle CSI = \angle TSI$.



Step 1. Let us prove $\angle RBI = \angle SCI$. We will use directed angles:

$$(BR, BI) = (BR, AB) + (AB, BI) = (AQ, AB) + (BI, BC)$$
$$= (CQ, BC) + (BI, BC) = (CI, CB) + (BI, BC),$$

which is symmetric in B, C. Therefore, analogously we would obtain the same expression for (CS, CI).

Step 2. Let us prove $BR/BI = CS/CI$. Clearly $BR = AQ$ and $CS = AP$. Angle chasing gives $\angle ICB = \angle QCB = \angle APQ$, and similarly $\angle PQA = \angle CBI$, and so $\triangle IBC \sim \triangle AQP$, from which the desired $AQ/BI = AP/CI$ follows. This finishes the solution.

Remark. In the alternative solutions below, the notation (ABC) refers to the circle passing through points A, B, and C. D will refer to the midpoint of arc BC not containing point A, unless stated otherwise.

Let I_A be the A-excenter of $\triangle ABC$. It is well-known that points B, C, I, and I_A lie on a circle with diameter II_A. The center of this circle is point D. It is clear that A, I and D are collinear.

In some of the solutions, some of the facts below may be used:

- B, T, I, C are concyclic. Indeed, $\angle BTC = \angle QAP = A + \frac{B+C}{2} = 90^\circ + \frac{A}{2} = \angle BIC$, so B, T, I, C are concyclic.
- $\angle QAP = \angle BTC = \angle RTS$. Indeed, since $QA \parallel BT$ and $PA \parallel CT$, then $\angle QAP = \angle BTC = \angle RTS$.
- RQ and PS are tangents to (ABC). Indeed, $\angle RQB = \angle QBA = \angle QAB$ and similarly for PS.
- $\triangle AQP \sim\ IBC$ (see Solution 1 for the proof).
- QP is the perpendicular bisector of AI. Indeed, $QA = QI$ and $PA = PI$ so $PQ \perp AI$ and PQ bisects AI.

Solution 2. We use complex numbers, with (ABC) as the unit circle. Set $D = d$, $P = p$, $Q = q$, so that $A = a = -\frac{pq}{d}$, $B = b = -\frac{dq}{p}$, and $C = c = -\frac{dp}{q}$. Write $z \sim w$ if z/w is a nonzero real number. We observe that

$$\frac{R-T}{S-T} \sim \frac{R-B}{S-C} = \frac{Q-A}{P-A} = \frac{q+\frac{pq}{d}}{p+\frac{pq}{d}} = \frac{(d+p)q}{(d+q)p}.$$

So, it suffices to show that $\frac{I-R}{I-S} \sim \frac{(d+p)q}{(d+q)p}$. Indeed,

$$I-R = (d+p+q)-(Q+B-A) = d+p+\frac{dq}{p}-\frac{pq}{d} = (d+p)\left(1+\frac{(d-p)q}{dp}\right) = \frac{(d+p)(dp+dq-pq)}{dp},$$

so

$$\frac{I-R}{I-S} = \frac{\frac{d+p}{dp}}{\frac{d+q}{dq}} = \frac{(d+p)q}{(d+q)p}.$$

Solution 3. In the following, all segment notations denote vectors.

As mentioned above, we find $\triangle AQP \sim \triangle IBC$, and by definitions of the parallelograms we have $BR = AQ$ and $CS = AP$ as well as $\angle RTS = \angle QAP$, so it suffices to show $\angle RIS = \angle QAP$. From the similarity $\triangle AQP \sim \triangle IBC$, we have a spiral map λ such that $IB = \lambda AQ$ and $IC = \lambda AP$. It follows that $IR = IB + BR = (\lambda+1)AQ$ and $IS = IC + CS = (\lambda+1)AP$. Because $\lambda + 1$ is also a spiral map, we have $\triangle IRS \sim \triangle AQP$ and in particular $\angle RIS = \angle QAP$, as we wanted to show.

Remark. This solution is deeply related to the complex numbers solution; indeed, the vectors can be interpreted as complex numbers and the spiral map as a complex scalar multiplication. But it only relies on the additive structure of the complex numbers as a real plane and the linear map acting on them (rather than, e.g., multiplying two points together), making vectors a slightly more natural language for the claims.

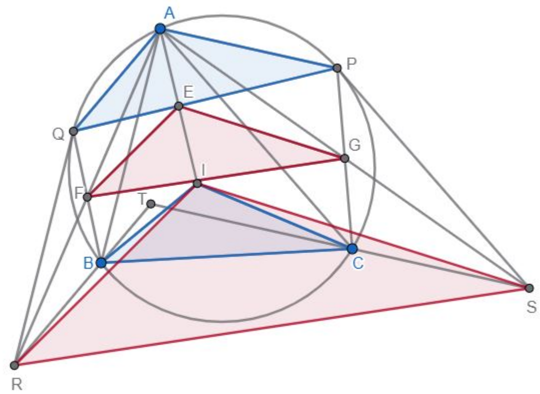
Remark. The number 1 in the solution above represents the identity map.

Solution 4. Let E, F, G be the midpoints of AI, BQ, CP. As in **Solution 1**, angle chase shows that $\triangle AQP \sim \triangle IBC$.

Note that by the Mean Geometry Theorem we have that $\frac{1}{2}AQP + \frac{1}{2}IBC = EFG$ is similar to $\triangle IBC$. Homothety with center A and scale-factor 2 maps EFG to IRS. Hence $\angle RIS = \angle FEG = \angle QAP = \angle BTC = \angle RTS$, so R, T, I, S are concyclic.

Remark. As shown above, E lies on QP and $AI \perp PQ$. One can prove that $\angle FEG = \angle BIC$ in another way. Let J be the midpoint of PQ. Then $\angle BIC = \angle FJG$ by midlines and $\angle FJG = \angle FEG$ by the lemma below applied in $BCPQ$.

Lemma. *Let $ABCD$ is a cyclic quadrilateral and E is the intersection of its diagonals. Then the midpoints of AB, BC, CD and the foot of the perpendicular from E to BC are concyclic.*



Solution 5. Let O be the circumcenter of (ABC). Let M, N, and L be the midpoints of OD, PC, and QB, respectively.

Claim 1. $\triangle OPQ$ and $\triangle DCB$ are directly similar.

Proof. Clearly $DB = DC$ and $OQ = OP$. Also note that $\angle QOP = 2\angle QDP = 2\angle QDA + 2\angle PDA = \angle BDA + \angle CDA = \angle BDC$. So the two triangles are directly similar by SAS.

Claim 2. $ML = MN$ and $\angle LMN = 180° - \angle BAC$.

Proof. Note that since $\triangle OQP \sim \triangle DBC$ by the Mean Geometry Theorem, we have that the average of the two triangles is also similar to them, therefore $\triangle MLN \sim \triangle DBC \Rightarrow ML = MN$ and $\angle LMN = \angle BDC = 180° - \angle BAC$.

Let K be the reflection of A over M.

Claim 3. K is the circumcenter of $\triangle RTS$.

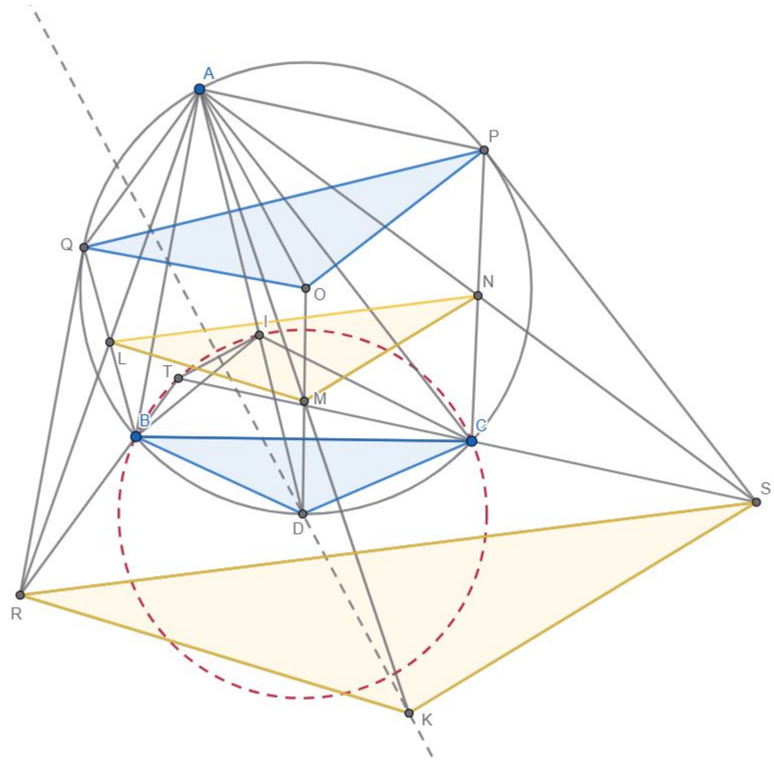
Proof. Note that since $AQRB$ and $APSC$ are parallelograms we have that $A - L - R$ are collinear and that $A - N - S$ are collinear. The homothety centered at A with scale-factor 2 maps $\triangle LMN$ to $\triangle RKS$, therefore $KR = KS$ and $\angle RKS = \angle LMN = \angle BDC = 2(180° - \angle RTS)$ (and K and T are in opposite sides of RS), implying that K is the circumcenter of $\triangle RTS$.

Claim 4. $KT = KI$.

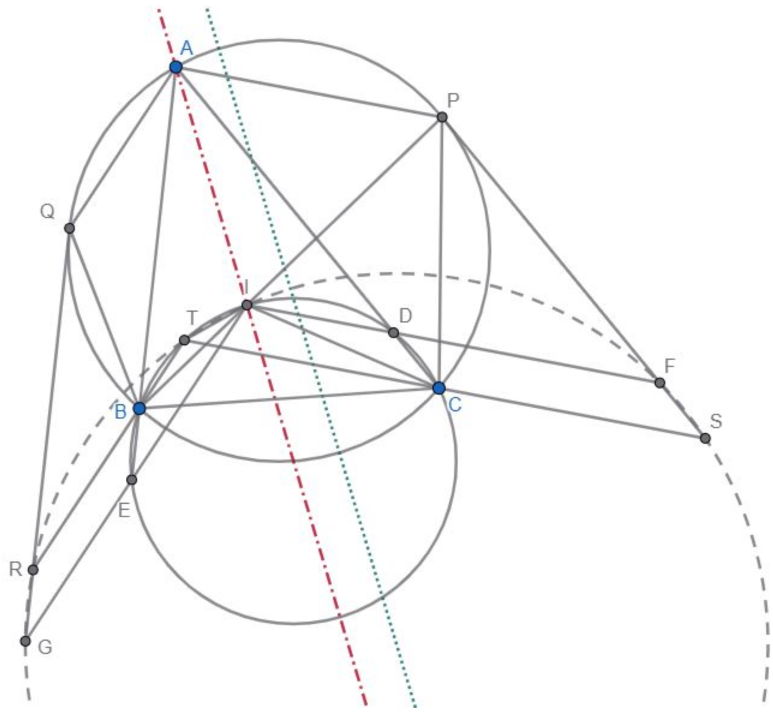
Proof. Note that $AOKD$ is a parallelogram. Let BT intersect the (ABC) again at point G. Since $\angle ABG = \angle ABT = \angle QAB = \angle QCA \Rightarrow AQ = AG$ and also $OQ = OG$ hence $AO \perp QG$. Then by Reim's theorem we have that $QG \parallel TI$ and also that $AO \parallel DK$, so $DK \perp TI$. Since $DI = DT$, it means that KD is the perpendicular bisector of TI, therefore $KT = KI$.

This means that $RTIS$ is cyclic with center K.

Remark. When K and T are on the same side of RS, it can be shown that $\angle RKS = 2\angle RTS$.

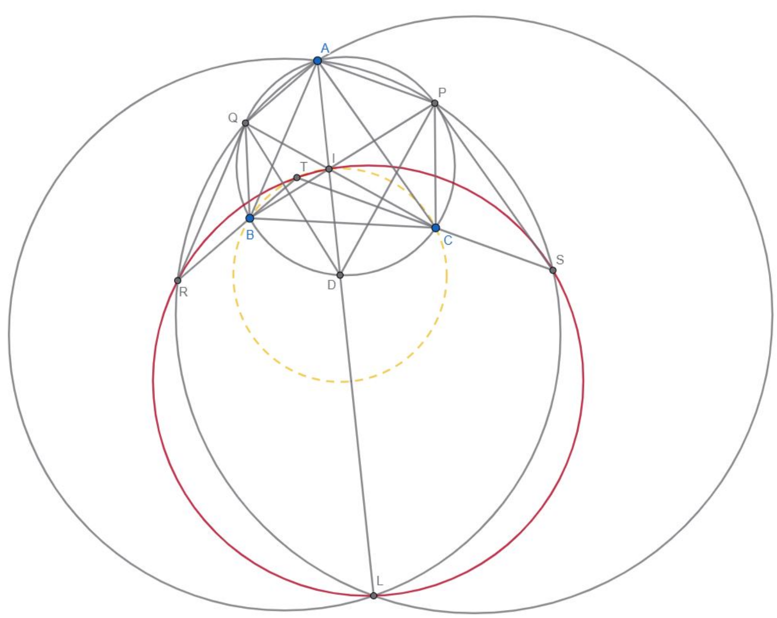


Solution 6. As shown above, we have that $BTIC$ is cyclic. Let D and E be the second intersections of AC and AB with this circle, respectively. Since the center of this circle lies on AI (by symmetry about AI), we have that $AB = AD$ and $AC = AE$, therefore $BE = CD$. Note that since $C - I - Q$ and $A - B - E$ are collinear, by Reim's theorem we have that $AQ \parallel EI$ and since $AQ \parallel BT$, we have that $BT \parallel EI$. Similarly, we get $CT \parallel DI$. Let F and G be the intersections of ID and IE with PS and QR, respectively. Clearly, $RGEB$ and $FSCD$ are parallelograms. Since $RGEB$ is parallelogram and $BEIT$ is isosceles trapezoid, we have that $RGIT$ is isosceles trapezoid. Similarly, $SFIT$ is isosceles trapezoid. Hence, both of them are cyclic. Note also that $QR = AB = AD = PF$ and $QG = AE = AC = PS$. Since QR and PS are tangents to the circumcircle of $\triangle ABC$ we have that R and F are symmetric (reflections) about the perpendicular bisector of PQ. Similarly, G and S are symmetric about the perpendicular bisector of PQ. This gives us that $QP \parallel RF \parallel GS$ and that $RFSG$ is an isosceles trapezoid, hence a cyclic quadrilateral with $\angle RGS = 180^\circ - \angle GQP = 180^\circ - \angle QAP = 180^\circ - \angle RTS \Rightarrow R, G, S, T$ are concylic. Combining all the facts about the cyclic quadrilaterals we proved above, we have that R, G, S, F, I, T are concylcic. Therefore R, T, I, S lie on a circle.

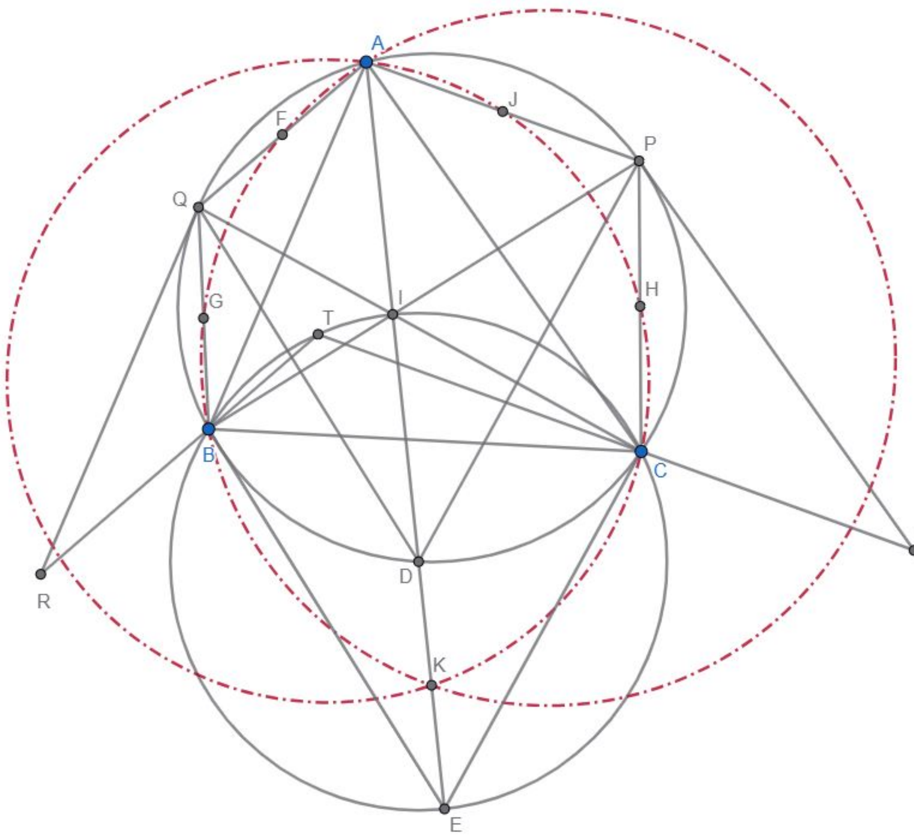


Solution 7. Let E be the A-excenter of $\triangle ABC$. Let the midpoints of AQ, QB, CP, PA be the points F, G, H, J, respectively. Both PD and EC are perpendicular to CI, hence $PD \parallel CE$.

Since $PA = PC$ we have that $AJHC$ is an isosceles trapezoid so it is cyclic. Let K be the second intersection of $(AJHC)$ and AI. Then $\angle ADP = \angle ACP = \angle ACH = \angle AKH \Rightarrow DP \parallel KH$. So KH is a line passing through the midpoint of the side CP of trapezoid $DPCE$ and parallel to the bases, hence K is the midpoint of DE. Similarly, we show that the circle $(AFGB)$ passes through the midpoint of DE. Homothety centered at A with scale-factor 2 maps (AJH) to (APS), (AFG) to (AQR), and line AI to line AI. This means that the circles (AQR) and (APS) intersect on AI, call it point L.



Now, $\angle ILR = 180^\circ - \angle AQR = \angle QAB = \angle QCB = 180^\circ - \angle ITB = 180^\circ - \angle ITR$, therefore R, L, I, T are concyclic. Similarly, we get that S, L, T, I are concyclic. Combining these, it means that R and S belong to the circle (LIT). The conclusion follows.



P5. Let $n > 1$ be an integer. In a *configuration* of an $n \times n$ board, each of the n^2 cells contains an arrow, either pointing up, down, left, or right. Given a starting configuration, Turbo the snail starts in one of the cells of the board and travels from cell to cell. In each move, Turbo moves one square unit in the direction indicated by the arrow in her cell (possibly leaving the board). After each move, the arrows in all of the cells rotate 90° counterclockwise. We call a cell *good* if, starting from that cell, Turbo visits each cell of the board exactly once, without leaving the board, and returns to her initial cell at the end. Determine, in terms of n, the maximum number of good cells over all possible starting configurations.

Solution. We will show that the maximum number of good cells over all possible starting configurations is

$$\frac{n^2}{4} \quad \text{if } n \text{ is even and}$$
$$0 \quad \text{if } n \text{ is odd.}$$

Odd n

First, we will prove that there are no good cells if n is an odd number.

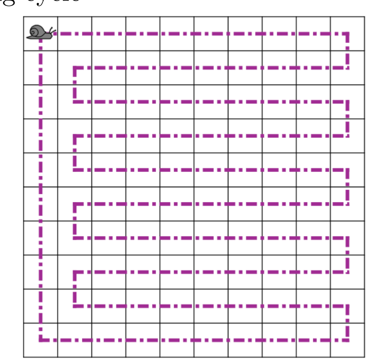
For Turbo to reach her goal, she must return to her initial cell after visiting every cell exactly once. Consider the chessboard coloring of the board. Without loss of generality, we assume that Turbo starts in a black cell. Since, at every step, Turbo moves to a cell of a different color; she will be in a white cell after $n^2 \equiv 1 \mod 2$ moves. Thus, it is impossible for Turbo to come back to her initial black cell on her n^2-th move, which is a contradiction. Thus there are no good cells.

Lower bound for even n

We will now construct a starting configuration with $\frac{n^2}{4}$ good cells for even n.

Let (i, j) denote the cell in row i and column j. Consider the following cycle

$$\begin{aligned}
(1,1) &\to (1,2) \to (1,3) \to \dots \to (1,n)\\
&\to (2,n) \to (2,n-1) \to \dots \to (2,2)\\
&\dots\\
&\to (2i-1,2) \to (2i-1,3) \to \dots \to (2i-1,n)\\
&\to (2i,n) \to (2i,n-1) \to \dots \to (2i,2)\\
&\dots\\
&\to (n,n) \to (n,n-1) \to \dots \to (n,2)\\
&\to (n,1) \to (n-1,1) \to \dots \to (2,1) \to (1,1).
\end{aligned}$$



Note that the cycle returns to the initial cell after visiting every cell exactly once. To prove that $(1, 1)$ is good, we need to find a starting configuration such that Turbo traverses this cycle.

Let c_i be the $(i-1)$-th cell on the cycle: so we have $c_0 = (1,1)$, $c_2 = (1,2)$, $\dots$, $c_{n^2-1} = (2,1)$. For every i, we draw an arrow in cell c_i pointing towards cell c_{i+1} (or pointing towards c_0 if $i = n^2 - 1$) and then rotate this arrow i times 90° in the clockwise direction. After i moves, the arrow in c_i will have rotated i times 90° counterclockwise and be in the same direction as on the path defined above. Thus, Turbo will traverse the cycle $c_0, c_1, c_2, ..., c_{n^2-1}, c_0$ and $(1, 1)$ is good.

Every four moves, all arrows point in the same direction as in the beginning. Moreover, the board will return to its initial configuration after traversing the full cycle, since n^2, the length of the cycle, is divisible by 4. Therefore Turbo can also start at any c_i with $4 \mid i$ and follow the same route. Hence the cells $c_0, c_4, c_8, \dots, c_{n^2-4}$ are good and there are $\frac{n^2}{4}$ of such cells.

Upper bound for even n

We will prove that for even n and any start configuration there are at most $\frac{n^2}{4}$ good cells.

Let a_0 be a good cell. Let $a_0, a_1, a_2, \dots, a_{n^2-1}, a_{n^2} = a_0$ be the sequence of cells that Turbo visits when she starts at a_0. Now suppose there is another good cell b_0 and let $b_0, b_1, b_2, \dots, b_{n^2-1}, b_{n^2} = b_0$ be the sequence of cells that Turbo visits when she starts at b_0.

Note that, since $4 \mid n^2$, the arrows are back to their initial configuration after n^2 steps. Thus, if Turbo keeps walking after returning to her initial cell, she would just traverse the same cycle over and over again.

Consider the upper left corner of the board. With standard row and column numbering, the corner cell is $(1,1)$. This cell has only two neighbours, so both the a-route and the b-route must have cells $(2,1),(1,1),(1,2)$ in that order or $(1,2),(1,1),(2,1)$ in that order. Without loss of generality, $a_{i-1} = (2,1)$, $a_i = (1,1)$ and $a_{i+1} = (1,2)$ for some i. Let j be such that $b_j = (1,1)$. If $b_{j-1} = (2,1) = a_{i-1}$, then the arrow in cell $(2,1)$ must be pointed in the same direction after $i-1$ steps and after $j-1$ steps, so $i \equiv j \mod 4$. But then the arrow in cell $b_j = (1,1) = a_i$ must also be pointed in the same direction after i and after j steps, so Turbo moves to $b_{j+1} = a_{i+1}$ in both cases, and again finds the arrow pointed in the same direction in both cases. Continuing, we find that the b-route is actually identical to $a_{4t}, a_{4t+1}, \dots, a_{n^2} = a_0, a_1, \dots, a_{4t-1}, a_{4t}$ for some t, as any other starting point would have the arrows in the wrong direction initially.

Now suppose instead that $b_{j+1} = (2,1) = a_{i-1}$. Considering the a-route, the arrows in the upper left corner after $i-1$ steps must be like this:

$\downarrow$	$\leftarrow$	
$\uparrow$		

or

$\downarrow$	$\uparrow$	
$\uparrow$		

Considering the b-route instead, the arrows after $j-1$ steps must be like this:

$\leftarrow$	$\leftarrow$	
$\uparrow$		

or

$\leftarrow$	$\leftarrow$	
$\leftarrow$		

From the arrows in cell $(1,1)$ we see that $i \equiv j+1 \mod 4$. However, for the cells $(2,1)$ and $(1,2)$ this gives a contradiction.

We conclude that the only possible good cells are a_{4t} for $t = 0, 1, \dots, \frac{n^2}{4} - 1$, which gives at most $\frac{n^2}{4}$ good cells.

P6. In each cell of a 2025×2025 board, a nonnegative real number is written in such a way that the sum of the numbers in each row is equal to 1, and the sum of the numbers in each column is equal to 1. Define r_i to be the largest value in row i, and let $R = r_1 + r_2 + \cdots + r_{2025}$. Similarly, define c_i to be the largest value in column i, and let $C = c_1 + c_2 + \cdots + c_{2025}$.

What is the largest possible value of $\frac{R}{C}$?

Solution 1. *Answer:* $\frac{2025}{89}$.

In general, if the table is $m^2 \times m^2$, the answer is $\frac{m^2}{2m-1}$.

The example is as follows: label rows and columns from 1 to m^2, from top to bottom and left to right. For the first m columns, write $\frac{1}{m}$ in all squares whose coordinates have the same residue modulo m and place 0 everywhere else. For the remaining $m^2 - m$ columns, place $\frac{1}{m^2}$ everywhere. Then $R = m^2 \cdot \frac{1}{m} = m$, and $C = m \cdot \frac{1}{m} + (m^2 - m) \cdot \frac{1}{m^2} = 2 - \frac{1}{m}$. So the ratio is as claimed.

$\frac{1}{2}$	0	$\frac{1}{4}$	$\frac{1}{4}$
0	$\frac{1}{2}$	$\frac{1}{4}$	$\frac{1}{4}$
$\frac{1}{2}$	0	$\frac{1}{4}$	$\frac{1}{4}$
0	$\frac{1}{2}$	$\frac{1}{4}$	$\frac{1}{4}$

In particular, when $n := m^2 = 2025$, we get $\frac{2025}{89}$. Now we need to show that $\frac{R}{C} \leq \frac{n}{2\sqrt{n}-1}$.

For each row, select one cell having the largest value appearing in said row and colour it red. Then, without loss of generality, we may rearrange the columns such that the red cells appear in the first k columns from the left, and each such column contains at least one red cell, for some $k \leq n$.

For the j^{th} column, for all $1 \leq j \leq k$, let p_j and n_j denote the sum and number of red cells in it, respectively. We observe that c_j, the biggest number in the j^{th} column, is at least $\frac{p_j}{n_j}$, for all $1 \leq j \leq k$. For all other columns, the largest value they contain is at least $\frac{1}{n}$, as their sum is 1. Thus, $C \geq \frac{p_1}{n_1} + \frac{p_2}{n_2} + \cdots + \frac{p_k}{n_k} + \frac{n-k}{n}$.

We can also observe that $R = p_1 + p_2 + \cdots + p_k$.

Therefore we have to show that:

$$p_1 + p_2 + \cdots + p_k \leq \frac{n}{2\sqrt{n}-1} \cdot \left(\frac{p_1}{n_1} + \frac{p_2}{n_2} + \cdots + \frac{p_k}{n_k} + \frac{n-k}{n} \right). \qquad (*)$$

By construction, $n_1 + n_2 + \cdots + n_k = n$, and, as the numbers in every column are nonnegative, we see that $p_j \leq 1$ for every j. Also, since each number in a red cell is at least $\frac{1}{n}$, we also have $p_j \geq \frac{n_j}{n}$.

Since our inequality is linear in each p_j, it suffices to prove it when each variable equals one of its two critical values. By relabeling, we may assume that $p_j = \frac{n_j}{n}$ for $1 \leq j \leq t$, and $p_j = 1$ for $t+1 \leq j \leq k$, for an integer $0 \leq t \leq k$.

First, if $t = k$, we observe that $p_1 + p_2 + \cdots + p_k = \frac{n_1}{n} + \frac{n_2}{n} + \cdots + \frac{n_k}{n} = 1$, and that $\frac{p_1}{n_1} + \frac{p_2}{n_2} + \cdots + \frac{p_k}{n_k} = \frac{k}{n}$, so the inequality becomes $1 \leq \frac{n}{2\sqrt{n}-1}$, which is true.

From now on we may assume that $t < k$. We need to show that:

$$\frac{n_1 + \cdots + n_t}{n} + k - t \leq \frac{n}{2\sqrt{n}-1} \cdot \left(\frac{t}{n} + \frac{1}{n_{t+1}} + \cdots + \frac{1}{n_k} + \frac{n-k}{n} \right).$$

By Cauchy–Schwarz inequality we have that:

$$\frac{1}{n_{t+1}} + \cdots + \frac{1}{n_k} \geq \frac{(k-t)^2}{n_{t+1} + \cdots + n_k} = \frac{(k-t)^2}{n - (n_1 + \cdots + n_t)}. \qquad \text{(CS)}$$

Let $n_1 + \cdots + n_t = n \cdot q$, where $0 \le q < 1$. Thus, it is now enough to show that:

$$q + k - t \le \frac{n}{2\sqrt{n} - 1} \left(\frac{t}{n} + \frac{(k-t)^2}{n - nq} + \frac{n-k}{n} \right).$$

Let $k - t = \ell \ge 1$. The inequality becomes:

$$q + \ell \le \frac{1}{2\sqrt{n} - 1} \cdot \left(n - \ell + \frac{\ell^2}{1 - q} \right).$$

Rearranging this we get:

$$n + q + \frac{\ell^2}{1 - q} \ge 2(q + \ell)\sqrt{n}.$$

If $q = 0$ the inequality is trivially true by $AM - GM$. Suppose now that $0 < q < 1$. Then, by Cauchy-Schwarz, we have:

$$q + \frac{\ell^2}{1 - q} = \frac{q^2}{q} + \frac{\ell^2}{1 - q} \ge (q + \ell)^2.$$

It is therefore enough to show that $n + (q + \ell)^2 \ge 2(q + \ell)\sqrt{n}$, which is true by $AM - GM$, completing the proof.

Remark. Another way to show $n + q + \frac{\ell^2}{1-q} \ge 2(q + \ell)\sqrt{n}$ is to split it into two $AM - GM$'s, as:

$$\left(n(1 - q) + \frac{\ell^2}{1 - q} \right) + (nq + q) \ge 2\ell\sqrt{n} + 2q\sqrt{n}.$$

Solution 1′. We prove the main inequality $(*)$ in a slightly different manner. Instead of the strong lower bound $p_j \ge \frac{n_j}{n}$, we use the weaker, simpler and more immediate lower bound $p_j \ge 0$ (thus proving the inequality in a larger regime).

As in Solution 1, suppose $p_j = 0$ for $1 \le j \le t$ and $p_j = 1$ for $t + 1 \le j \le k$, with $\ell = k - t$. We also denote by $m = n_{t+1} + \cdots + n_k$ and note that $m \le n - t$, since $n_j \ge 1$ for each $i \le t$. We need to prove that:

$$\ell \le \frac{n}{2\sqrt{n} - 1} \cdot \left(\frac{1}{n_{t+1}} + \cdots + \frac{1}{n_k} + \frac{n-k}{n} \right).$$

Rearranging and using the same Cauchy-Schwarz (CS) as in Solution 1, we see it suffices to show that:

$$(2\sqrt{n} - 1)\ell \le n - k + \frac{n\ell^2}{m},$$

or equivalently, that:

$$2\sqrt{n}\ell \le n - t + \frac{n\ell^2}{m}.$$

But since $n - t \ge m$ this immediately follows from $2\sqrt{n}\ell \le m + \frac{n\ell^2}{m}$, which is a simple application of $AM - GM$.

(Note that in Solution 1 the case $t = k$, which is equivalent to $m = \ell = 0$, was dealt with separately, to avoid the appearance of $\frac{0}{0}$ terms such as $\frac{n\ell^2}{m}$. It is easy to verify that the corresponding term should in fact be 0 and the final $AM - GM$ replaced with $0 \le 0$, and all transitions are valid. Alternatively, the case can be argued directly by simply noting that $(*)$ evaluates to $0 \le \frac{n-k}{2\sqrt{n}-1}$, which is obvious.)

Solution 1″. This is an alternative way of getting the upper bound on $\frac{R}{C}$ from

$$\frac{R}{C} \le \frac{p_1 + p_2 + \ldots + p_k}{\frac{p_1}{n_1} + \frac{p_2}{n_2} + \ldots + \frac{p_k}{n_k} + \frac{n-k}{n}}.$$

Using the fact that $\sum_{j=1}^{k} n_j = n$, we can rewrite the above right hand side as follows:

$$\frac{\sum_{j=1}^{k} p_j}{\sum_{j=1}^{k}\left(\frac{p_j}{n_j}+\frac{n_j-1}{n}\right)}.$$

We notice that this is a quotient of affine functions in the p_j's, for which the denominator does not vanish over the set defined by $0 \le p_j \le 1$. Therefore the maximum of this function is attained when a certain number of p_j's are 1 and the others are 0. Without loss of generality we may assume that the first t are equal to 1 and the other $k-t$ are 0 for some $0 \le t \le k$. Then one has that the previous expression is at most

$$\frac{t}{\sum_{1\le j\le t}\left(\frac{1}{n_j}+\frac{n_j-1}{n}\right)+\sum_{t<j\le k}\frac{n_j-1}{n}}.$$

We now lower bound the denominator by observing that the second sum is non negative, while each term of the first sum can be bounded by $AM-GM$ as follows:

$$\frac{1}{n_j}+\frac{n_j-1}{n} \ge \frac{2}{\sqrt{n}}-\frac{1}{n}.$$

We therefore have

$$\frac{R}{C} \le \max_{1\le t\le k}\frac{t}{\sum_{1\le j\le t}(\frac{2}{\sqrt{n}}-\frac{1}{n})} = \frac{n}{2\sqrt{n}-1},$$

which finishes the proof.

Solution 1‴. We follow the same notation as above. First, we apply Cauchy-Schwarz as follows :

$$\left(\sum_{i=1}^{k}\frac{p_i}{n_i}\right)\left(\sum_{i=1}^{k}p_in_i\right) \ge \left(\sum_{i=1}^{k}p_i\right)^2 = R^2.$$

We now write $z_i = 1-p_i$ for all $1 \le i \le k$, and observe that all z_i are positive. Moreover, we have that $\sum_{i=1}^{k} z_i = k-R$, and $\sum_{i=1}^{k} p_in_i = n-\sum_{i=1}^{k} p_iz_i$. Thus, from our last inequality we get

$$\frac{p_1}{n_1}+\cdots+\frac{p_k}{n_k} \ge \frac{R^2}{n-\sum_{i=1}^{k} n_iz_i}.$$

As before we have $C \ge \frac{p_1}{n_1}+\cdots+\frac{p_k}{n_k}+\frac{n-k}{n}$, and so

$$C \ge \frac{R^2}{n-\sum_{i=1}^{k} n_iz_i}+\frac{n-k}{n} = \frac{R^2}{n-\sum_{i=1}^{k} n_iz_i}+\frac{n-R-\sum_{i=1}^{k} z_i}{n}.$$

Putting everything together we get

$$\frac{R}{C} \le \frac{R}{\frac{R^2}{n-\sum_{i=1}^{k} n_iz_i}+\frac{n-R-\sum_{i=1}^{k} z_i}{n}} = \frac{n}{\frac{Rn}{n-\sum_{i=1}^{k} n_iz_i}+\frac{n-\sum_{i=1}^{k} z_i}{R}-1}.$$

Applying $AM-GM$ to the denominator we get $\frac{R}{C} \le \frac{n}{2\sqrt{n\frac{n-\sum_{i=1}^{k} z_i}{n-\sum_{i=1}^{k} n_iz_i}}-1}$, which finishes the proof by noting that $\frac{n-\sum_{i=1}^{k} z_i}{n-\sum_{i=1}^{k} n_iz_i} \ge 1$.

Solution 2. This is an alternative approach that goes via an intermediary quantity in order to establish to the upper bound on R/C.

Let x_{ij} be the entry in row i and column j. Let $n = 2025$. The key idea is to analyze the expression:

$$T := \sum_{i,j} x_{ij} \max\left(x_{ij}, \frac{1}{n}\right).$$

On one hand, since $c_j \geq 1/n$, we have $x_{ij} \max(x_{ij}, \frac{1}{n}) \leq x_{ij} c_j$ for every (i, j). So

$$T \leq \sum_j \sum_i x_{ij} c_j = \sum_j c_j = C.$$

On the other hand, let j_i be one of the indices for which $r_i = x_{ij_i}$. We therefore have:

$$\begin{aligned}
T &= \sum_i \left(x_{ij_i} \max\left(x_{ij_i}, \frac{1}{n}\right) + \sum_{j \neq j_i} x_{ij} \max\left(x_{ij}, \frac{1}{n}\right)\right) \\
&\geq \sum_i \left(r_i^2 + \frac{1}{n} \sum_{j \neq j_i} x_{ij} \right) \\
&= \sum_i \left(r_i^2 - \frac{1}{n} r_i + \frac{1}{n} \right) \geq \sum_i \left(\frac{2}{\sqrt{n}} r_i - \frac{1}{n} r_i \right) = \left(\frac{2}{\sqrt{n}} - \frac{1}{n} \right) R.
\end{aligned}$$

This gives the claimed result.